International Journal of Heat and Mass Transfer 52 (2009) 5796–5801

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

Thermal instability in a porous medium layer saturated by a nanofluid

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ARTICLE INFO

Article history:
Received 18 March 2009
Received in revised form 17 July 2009
Accepted 17 July 2009
Available online 9 September 2009

Keywords:
Nanofluid
Porous medium
Instability
Natural convection

ABSTRACT

The onset of convection in a horizontal layer of a porous medium saturated by a nanofluid is studied analytically. The model used for the nanofluid incorporates the effects of Brownian motion and thermophoresis. The analysis reveals that for a typical nanofluid (with large Lewis number) the prime effect of the nanofluids is via a buoyancy effect coupled with the conservation of nanoparticles, the contribution of nanoparticles to the thermal energy equation being a second-order effect. It is found that the critical thermal Rayleigh number can be reduced or increased by a substantial amount, depending on whether the basic nanoparticle distribution is top-heavy or bottom-heavy, by the presence of the nanoparticles. Oscillatory instability is possible in the case of a bottom-heavy nanoparticle distribution.

1. Introduction

The term "nanofluid" refers to a liquid containing a dispersion of submicronic solid particles (nanoparticles). The term was coined by Choi [1]. The characteristic feature of nanofluids is thermal conductivity enhancement, a phenomenon observed by Masuda et al. [2]. This phenomenon suggests the possibility of using nanofluids in advanced nuclear systems [3]. Another recent application of nanofluid flow is nano-drug delivery [4].

A comprehensive survey of convective transport in nanofluids was made by Buongiorno [5], who says that a satisfactory explanation for the abnormal increase of the thermal conductivity and viscosity is yet to be found. He focused on the further heat transfer enhancement observed in convective situations. Buongiorno notes that several authors have suggested that convective heat transfer enhancement could be due to the dispersion of the suspended nanoparticles but he argues that this effect is too small to explain the observed enhancement. Buongiorno also concludes that turbulence is not affected by the presence of the nanoparticles so this cannot explain the observed enhancement. Particle rotation has also been proposed as a cause of heat transfer enhancement, but Buongiorno calculates that this effect is too small to explain the effect. With dispersion, turbulence and particle rotation ruled out as significant agencies for heat transfer enhancement, Buongiorno proposed a new model based on the mechanics of the nanoparticle/base-fluid relative velocity.

Buongiorno [5] noted that the nanoparticle absolute velocity can be viewed as the sum of the base-fluid velocity and a relative velocity (that he calls the slip velocity). He considered in turn seven slip mechanisms: inertia, Brownian diffusion, thermophoresis, diffusiophoresis, Magnus effect, fluid drainage and gravity settling. He concluded that in the absence of turbulent effects it is the Brownian diffusion and the thermophoresis that will be important. Buongiorno proceeded to write down conservation equations based on these two effects.

The Bénard problem (the onset of convection in a horizontal layer uniformly heated from below) for a nanofluid was studied by Tzou [6,7] on the basis of the transport equations of Buongiorno [5]. In the present paper the corresponding problem for flow in a porous medium (the Horton–Rogers–Lapwood problem) is studied. We will assume that nanoparticles are suspended in the nanofluid using either surfactant or surface charge technology. This prevents particles from agglomeration and deposition on the porous matrix.

For completeness, we mention that a substantially different treatment of the Bénard problem for a nanofluid has been given by Kim et al. [8–10]. These authors simply modified three quantities that appear in the definition of the Rayleigh number, namely the thermal expansion coefficient, the thermal diffusivity and the kinematic viscosity.

We are not aware of any publications on convection of nanofluids in porous media as such. (We are aware of the paper by Tsai and Chein [11] who modelled a microchannel heat sink, with a nanofluid, as a porous medium.) There have been studies done on convection in porous media with thermophoresis particle deposition (e.g., [12]) but an essential feature of nanofluids is that with a special treatment particle deposition can be made negligible.

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doi:10.1016/j.ijheatmasstransfer.2009.07.023

Nomenclature

c	nanofluid specific heat at constant pressure
c_p	specific heat of the nanoparticle material
$(\rho c)_m$	effective heat capacity of the porous medium
d_p	nanoparticle diameter
D_B	Brownian diffusion coefficient, given by Eq. (4)
D_T	thermophoretic diffusion coefficient, given by Eq. (8)
h_p	specific enthalpy of the nanoparticle material
H	dimensional layer depth
$\mathbf{j}_p$	diffusion mass flux for the nanoparticles, given by Eq. (3)
$\mathbf{j}_{p,T}$	thermophoretic diffusion, given by Eq. (7)
k	thermal conductivity of the nanofluid
k_B	Boltzmann's constant
k_m	effective thermal conductivity of the porous medium
k_p	thermal conductivity of the particle material
Le	Lewis number, defined by Eq. (34)
N_A	modified diffusivity ratio, defined by Eq. (38)
N_B	modified particle-density increment, defined by Eq. (39)
p^*	pressure
p	dimensionless pressure, $p^*K/\mu\alpha_m$
$\mathbf{q}$	energy flux relative to a frame moving with the nanofluid velocity $\mathbf{v}$
Ra	thermal Rayleigh–Darcy number, defined by Eq. (35)
Rm	basic-density Rayleigh number, defined by Eq. (36)
Rn	concentration Rayleigh number, defined by Eq. (37)
t^*	time
t	dimensionless time, $t^*\alpha_m/\sigma H^2$
T^*	nanofluid temperature
T	dimensionless temperature, $\frac{T^*-T_c^*}{T_h^*-T_c^*}$
T_c^*	temperature at the upper wall
T_h^*	temperature at the lower wall
(u, v, w)	dimensionless Darcy velocity components $(u^*, v^*, w^*) \times H/\alpha_m$
$\mathbf{v}$	nanofluid velocity
$\mathbf{v}_D$	Darcy velocity $\varepsilon\mathbf{v}$
$\mathbf{v}_D^*$	dimensional Darcy velocity (u^*, v^*, w^*)
$\mathbf{V}_T$	thermophoretic velocity
(x, y, z)	dimensionless Cartesian coordinates $(x^*, y^*, z^*)/H$; z is the vertically upward coordinate
(x^*, y^*, z^*)	Cartesian coordinates
Greek symbols	
α_m	thermal diffusivity of the porous medium, $\frac{k_m}{(\rho c_P)_f}$
$\tilde{\beta}$	proportionality factor, given by Eq. (6)
μ	viscosity of the fluid
$\tilde{\mu}$	effective viscosity of the porous medium
ρ	fluid density
ρ_p	nanoparticle mass density
σ	parameter defined by Eq. (27)
ϕ^*	nanoparticle volume fraction
ϕ	relative nanoparticle volume fraction, $\frac{\phi^*-\phi_0^*}{\phi_1^*-\phi_0^*}$
Superscripts	
$*$	dimensional variable (used from Section 4 onwards)
$'$	perturbation variable
Subscript	
b	basic solution

Likewise it appears that studies involving Brownian motion and porous media are confined to deposition phenomena and so are irrelevant to the present investigation.

The present authors [13] have written a companion paper on the Rayleigh–Bénard problem.

2. Conservation equations for a nanofluid

First, we outline the derivation of conservation equations applicable to a nanofluid in the absence of a solid matrix. Later we modify these equations to the case of a porous medium saturated by the nanofluid.

The Buongiorno model treats the nanofluid as a two-component mixture (base fluid plus nanoparticles) with the following assumptions:

1. incompressible flow,
2. no chemical reactions,
3. negligible external forces,
4. dilute mixture,
5. negligible viscous dissipation,
6. negligible radiative heat transfer,
7. nanoparticles and base fluid locally in thermal equilibrium.

In Sections 2 and 3, all the variables are dimensional. The continuity equation for the nanofluid is

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

Here $\mathbf{v}$ is the nanofluid velocity.

The conservation equation for the nanoparticles in the absence of chemical reactions is

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla\phi = -\frac{1}{\rho_p}\nabla \cdot \mathbf{j}_p, \tag{2}$$

where ϕ is nanoparticle volume fraction, ρ_p is the nanoparticle mass density and $\mathbf{j}_p$ is the diffusion mass flux for the nanoparticles, given as the sum of two diffusion terms (Brownian diffusion and thermophoresis) by

$$\mathbf{j}_p = \mathbf{j}_{p,B} + \mathbf{j}_{p,T} = -\rho_p D_B \nabla\phi - \rho_p D_T \frac{\nabla T}{T}. \tag{3}$$

(Thermophoresis is the "particle" equivalent of the Soret effect in gaseous or liquid mixtures.)

Here D_B is the Brownian diffusion coefficient given by the Einstein–Stokes equation

$$D_B = \frac{k_B T}{3\pi\mu d_p}, \tag{4}$$

where k_B is the Boltzmann's constant, μ is the viscosity of the fluid and d_p is the nanoparticle diameter. Use has been made of the expression

$$\mathbf{V}_T = -\tilde{\beta}\frac{\mu}{\rho}\frac{\nabla T}{T} \tag{5}$$

for the thermophoretic velocity $\mathbf{V}_T$. Here ρ is the fluid density and the proportionality factor $\tilde{\beta}$ is given by

$$\tilde{\beta} = 0.26\frac{k}{2k + k_p}, \tag{6}$$

where k and k_p are the thermal conductivities of the fluid and the particle material. Hence the thermophoretic diffusion flux is given by

$$\mathbf{j}_{p,T} = \rho_p \phi \mathbf{V}_T = -\rho_p D_T \frac{\nabla T}{T}, \tag{7}$$

where the thermophoretic diffusion coefficient is given by

$$D_T = \tilde{\beta}\frac{\mu}{\rho}\phi. \tag{8}$$

Eqs. (2) and (3) then produce the conservation equation in the form

$$\frac{\partial \phi}{\partial t} + \mathbf{v}\cdot\nabla\phi = \nabla\cdot\left[D_B\nabla\phi + D_T\frac{\nabla T}{T}\right]. \tag{9}$$

The momentum equation for a nanofluid takes the same form as for a pure fluid, but it should be remembered that μ is a strong function of ϕ. If one introduces a buoyancy force and adopts the Boussinesq approximation, then the momentum equation can be written as

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}, \tag{10}$$

where

$$\rho = \phi\rho_p + (1-\phi)\rho_f. \tag{11}$$

The nanofluid density ρ can be approximated by the base-fluid density ρ_f when ϕ is small. Then, when the Boussinesq approximation is adopted the buoyancy term is approximated by

$$\rho\mathbf{g} \cong [\phi\rho_p + (1-\phi)\{\rho(1-\beta(T-T_0))\}]\mathbf{g}. \tag{12}$$

The thermal energy equation for a nanofluid can be written as

$$\rho c\left[\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T\right] = -\nabla\cdot\mathbf{q} + h_p\nabla\cdot\mathbf{j}_p, \tag{13}$$

where c is the nanofluid specific heat, T is the nanofluid temperature, h_p is the specific enthalpy of the nanoparticle material and $\mathbf{q}$ is the energy flux, relative to a frame moving with the nanofluid velocity $\mathbf{v}$, given by

$$\mathbf{q} = -k\nabla T + h_p\mathbf{j}_p, \tag{14}$$

where k is the nanofluid thermal conductivity. Substituting Eq. (14) in Eq. (13) yields

$$\rho c\left[\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T\right] = \nabla\cdot(k\nabla T) - c_p\mathbf{j}_p\cdot\nabla T. \tag{15}$$

In deriving this equation use has been made of a vector identity and the fact (deriving from assumption (7)) that $\nabla h_p = c_p\nabla T$, where c_p is the nanoparticle specific heat of the material constituting the nanoparticles, while c is the specific heat (at constant pressure) of the fluid. Then substitution of Eq. (3) in Eq. (15) gives the final form

$$\rho c\left[\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T\right] = \nabla\cdot(k\nabla T) + \rho_p c_p\left[D_B\nabla\phi\cdot\nabla T + D_T\frac{\nabla T\cdot\nabla T}{T}\right]. \tag{16}$$

3. Conservation equations for a porous medium saturated by a nanofluid

We consider a porous medium whose porosity is denoted by ε and permeability by K. A subscript s will now be used to denote properties of the solid matrix. The Darcy velocity is denoted by $\mathbf{v}_D$. This is related to $\mathbf{v}$ by $\mathbf{v}_D = \varepsilon\mathbf{v}$. We now have to deal with the following four field equations (corresponding to Eqs. (1), (10), (16), (9)), for total mass, momentum, thermal energy and nanoparticles, respectively

$$\nabla\cdot\mathbf{v}_D = 0, \tag{17}$$

$$\rho\left(\frac{1}{\varepsilon}\frac{\partial \mathbf{v}_D}{\partial t} + \frac{1}{\varepsilon^2}\mathbf{v}_D\cdot\nabla\mathbf{v}_D\right) = -\nabla p + \tilde{\mu}\nabla^2\mathbf{v}_D - \frac{\mu}{K}\mathbf{v}_D + [\phi\rho_p + (1-\phi)\{\rho(1-\beta(T-T_0))\}]\mathbf{g}, \tag{18}$$

$$(\rho c)_m\frac{\partial T}{\partial t} + (\rho c)_f\mathbf{v}_D\cdot\nabla T = \nabla\cdot(k_m\nabla T) + \varepsilon(\rho c)_p\left[D_B\nabla\phi\cdot\nabla T + D_T\frac{\nabla T\cdot\nabla T}{T}\right], \tag{19}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{\varepsilon}\mathbf{v}_D\cdot\nabla\phi = \nabla\cdot\left[D_B\nabla\phi + D_T\frac{\nabla T}{T}\right]. \tag{20}$$

Here we have introduced the effective viscosity $\tilde{\mu}$, the effective heat capacity $(\rho c)_m$ and the effective thermal conductivity k_m of the porous medium.

In deriving Eqs. (17)–(20) we have assumed that the Brownian motion and thermophoresis processes remain coherent while volume averages over a representative elementary volume are taken. This assumption can be questioned. In the context of modelling transport in porous media, Eqs. (17) and (18) are standard. Eq. (20) involves just intrinsic quantities in the sense that the average is being taken over the nanofluid only and the solid matrix is not involved. The question thus reduces to whether the terms within the square brackets on the right-hand side of Eq. (19) need modification. We recall that in nanofluids the particles are so small that for practical purposes they remain in suspension in a uniform manner. We emphasize our assumption that the nanoparticles are suspended in nanofluid using either surfactant or surface charge technology, something that prevents particles from agglomeration and deposition on the porous matrix. We suggest that then it is reasonable to assume as a first approximation that no modification to Eq. (19) is necessary.

4. Application to the Horton–Rogers–Lapwood problem

We select a coordinate frame in which the z-axis is aligned vertically upwards. We consider a horizontal layer of a porous medium confined between the planes $z^* = 0$ and $z^* = H$. From now on asterisks are used to denote dimensional variables (previously an asterisk has not been needed because all the variables were dimensional). Each boundary wall is assumed to be impermeable and perfectly thermally conducting. The temperatures at the lower and upper wall are taken to be T_h^* and T_c^*, the former being the greater. For simplicity, Darcy's law is assumed to hold and the Oberbeck–Boussinesq approximation is employed. Homogeneity and local thermal equilibrium in the porous medium are assumed. The reference temperature is taken to be T_c^*. In the linear theory being applied here the temperature change in the fluid is assumed to be small in comparison with T_c^*. Eqs. (18)–(20) take the form

$$0 = -\nabla^* p^* - \frac{\mu}{K}\mathbf{v}_D^* + \left[\phi^*\rho_p + (1-\phi^*)\{\rho(1-\beta(T^*-T_c^*))\}\right]\mathbf{g}, \tag{21}$$

$$(\rho c)_m\frac{\partial T^*}{\partial t^*} + (\rho c)_f\mathbf{v}_D^*\cdot\nabla T^* = k_m\nabla^{*2}T^* + \varepsilon(\rho c)_p[D_B\nabla^*\phi^*\cdot\nabla^*T^* + \left(D_T/T_c^*\right)\nabla^*T^*\cdot\nabla^*T^*], \tag{22}$$

$$\frac{\partial \phi^*}{\partial t^*} + \frac{1}{\varepsilon}\mathbf{v}_D^*\cdot\nabla^*\phi^* = D_B\nabla^{*2}\phi^* + \left(D_T/T_c^*\right)\nabla^{*2}T^*. \tag{23}$$

We write $\mathbf{v}_D^* = (u^*, v^*, w^*)$.

We assume that the temperature and the volumetric fraction of the nanoparticles are constant on the boundaries. Thus the boundary conditions are

$$w^* = 0, \quad T^* = T_h^*, \quad \phi^* = \phi_0^* \quad \text{at} \quad z^* = 0, \tag{24}$$

$$w^* = 0, \quad T^* = T_c^*, \quad \phi^* = \phi_1^* \quad \text{at} \quad z^* = H. \tag{25}$$

We recognize that our choice of boundary conditions imposed on ϕ^* is somewhat arbitrary. It could be argued that zero particle flux on the boundaries is more realistic physically, but then one is faced with the problem that it appears that no steady-state solution for the basic conduction equations is then possible, so that in order to make analytical progress it is necessary to freeze the basic profile for ϕ^*, and at that stage our choice of boundary conditions is seen to be quite realistic.

We introduce dimensionless variables as follows. We define

$$(x,y,z) = (x^*,y^*,z^*)/H, \quad t = t^*\alpha_m/\sigma H^2,$$

$$(u,v,w) = (u^*,v^*,w^*)H/\alpha_m, \quad p = p^*K/\mu\alpha_m,$$

$$\phi = \frac{\phi^* - \phi_0^*}{\phi_1^* - \phi_0^*}, \quad T = \frac{T^* - T_c^*}{T_h^* - T_c^*}, \tag{26}$$

where

$$\alpha_m = \frac{k_m}{(\rho c_P)_f}, \quad \sigma = \frac{(\rho c_P)_m}{(\rho c_P)_f}. \tag{27}$$

Then Eq. (17) and (21)–(25) take the form:

$$\nabla \cdot \mathbf{v} = 0, \tag{28}$$

$$0 = -\nabla p - \mathbf{v} - \mathrm{Rm}\hat{\mathbf{e}}_z + \mathrm{Ra}T\hat{\mathbf{e}}_z - \mathrm{Rn}\phi\hat{\mathbf{e}}_z, \tag{29}$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \nabla^2 T + \frac{N_B}{\mathrm{Le}}\nabla\phi\cdot\nabla T + \frac{N_A N_B}{\mathrm{Le}}\nabla T\cdot\nabla T, \tag{30}$$

$$\frac{1}{\sigma}\frac{\partial\phi}{\partial t} + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla\phi = \frac{1}{\mathrm{Le}}\nabla^2\phi + \frac{N_A}{\mathrm{Le}}\nabla^2 T, \tag{31}$$

$$w = 0, \quad T = 1, \quad \phi = 0 \quad \text{at} \quad z = 0, \tag{32}$$

$$w = 0, \quad T = 0, \quad \phi = 1 \quad \text{at} \quad z = 1. \tag{33}$$

Here

$$\mathrm{Le} = \frac{\alpha_m}{D_B}, \tag{34}$$

$$\mathrm{Ra} = \frac{\rho g\beta KH(T_h^* - T_c^*)}{\mu\alpha_m}, \tag{35}$$

$$\mathrm{Rm} = \frac{\left[\rho_p\phi_1^* + \rho(1-\phi_1^*)\right]gKH}{\mu\alpha_m}, \tag{36}$$

$$\mathrm{Rn} = \frac{(\rho_p - \rho)(\phi_1^* - \phi_0^*)gKH}{\mu\alpha_m}, \tag{37}$$

$$N_A = \frac{D_T(T_h^* - T_c^*)}{D_B T_c^*(\phi_1^* - \phi_0^*)}, \tag{38}$$

$$N_B = \frac{\varepsilon(\rho c)_p}{(\rho c)_f}(\phi_1^* - \phi_0^*). \tag{39}$$

The parameter Le is a Lewis number and Ra is the familiar thermal Rayleigh–Darcy number. The new parameters Rm and Rn may be regarded as a basic-density Rayleigh number and a concentration Rayleigh number, respectively. The parameter N_A is a modified diffusivity ratio and is somewhat similar to the Soret parameter that arises in cross-diffusion phenomena in solutions, while N_B is a modified particle-density increment.

In the spirit of the Oberbeck–Boussinesq approximation, Eq. (29) has been linearized by the neglect of a term proportional to the product of ϕ and T. This assumption is likely to be valid in the case of small temperature gradients in a dilute suspension of nanoparticles.

4.1. Basic solution

We seek a time-independent quiescent solution of Eqs. (28)–(33) with temperature and nanoparticle volume fraction varying in the z-direction only, that is a solution of the form

$$\mathbf{v} = 0, \quad T = T_b(z), \quad \phi = \phi_b(z).$$

Eqs. (30) and (31) reduce to

$$\frac{d^2T_b}{dz^2} + \frac{N_B}{\mathrm{Le}}\frac{d\phi_b}{dz}\frac{dT_b}{dz} + \frac{N_A N_B}{\mathrm{Le}}\left(\frac{dT_b}{dz}\right)^2 = 0, \tag{40}$$

$$\frac{d^2\phi_b}{dz^2} + N_A\frac{d^2T_b}{dz^2} = 0. \tag{41}$$

Using the boundary conditions (32) and (33), Eq. (41) may be integrated to give

$$\phi_b = -N_A T_b + (1-N_A)z + N_A, \tag{42}$$

and substitution of this into Eq. (40) gives

$$\frac{d^2T_b}{dz^2} + \frac{(1-N_A)N_B}{\mathrm{Le}}\frac{dT_b}{dz} = 0. \tag{43}$$

The solution of Eq. (43) satisfying Eqs. (32) and (33) is

$$T_b = \frac{1 - e^{-(1-N_A)N_B(1-z)/\mathrm{Le}}}{1 - e^{-(1-N_A)N_B/\mathrm{Le}}}. \tag{44}$$

The remainder of the basic solution is easily obtained by first substituting in Eq. (42) to obtain ϕ_b and then using integration of Eq. (29) to obtain p_b.

According to Buongiorno [5], for most nanofluids investigated so far $\mathrm{Le}/(\phi_1^* - \phi_0^*)$ is large, of order 10^5–10^6, and since the nanoparticle fraction decrement $(\phi_1^* - \phi_0^*)$ is typically no smaller than 10^{-3} this means so that Le is large, of order 10^2–10^3, while N_A is no greater than about 10. Then the exponents in Eqs. (43) and (44) are small and so to a good approximation one has

$$T_b = 1 - z, \tag{45}$$

and so

$$\phi_b = z. \tag{46}$$

4.2. Perturbation solution

We now superimpose perturbations on the basic solution. We write

$$\mathbf{v} = \mathbf{v}', \quad p = p_b + p', \quad T = T_b + T', \quad \phi = \phi_b + \phi', \tag{47}$$

substitute in Eqs. (28)–(33), and linearize by neglecting products of primed quantities. The following equations are obtained when Eqs. (45) and (46) are used.

$$\nabla\cdot\mathbf{v}' = 0, \tag{48}$$

$$0 = -\nabla p' - \mathbf{v}' + \mathrm{Ra}T'\hat{\mathbf{e}}_z - \mathrm{Rn}\phi'\hat{\mathbf{e}}_z, \tag{49}$$

$$\frac{\partial T'}{\partial t} - w' = \nabla^2 T' + \frac{N_B}{\mathrm{Le}}\left(\frac{\partial T'}{\partial z} - \frac{\partial\phi'}{\partial z}\right) - \frac{2N_A N_B}{\mathrm{Le}}\frac{\partial T'}{\partial z}, \tag{50}$$

$$\frac{1}{\sigma}\frac{\partial\phi'}{\partial t} + \frac{1}{\varepsilon}w' = \frac{1}{\mathrm{Le}}\nabla^2\phi' + \frac{N_A}{\mathrm{Le}}\nabla^2 T', \tag{51}$$

$$w' = 0, \quad T' = 0, \quad \phi' = 0 \quad \text{at} \quad z = 0 \text{ and at } z = 1. \tag{52}$$

It will be noted that the parameter Rm is not involved in these and subsequent equations. It is just a measure of the basic static pressure gradient.

For the case of a regular fluid (not a nanofluid) the parameters Rn, N_A and N_B are zero, the second term in Eq. (51) is absent because $d\phi_b/dz = 0$ and then Eq. (51) is satisfied trivially. The remaining equations are reduced to the familiar equations for the Horton–Roger–Lapwood problem.

The six unknowns $u', v', w', p', T', \phi'$ can be reduced to three by operating on Eq. (49) with $\hat{\mathbf{e}}_z\cdot$ *curl curl* and using the identity *curl curl* $\equiv$ *grad div* $-\nabla^2$ together with Eq. (48).

The result is

$$\nabla^2 w' = \mathrm{Ra}\nabla_H^2 T' + \mathrm{Rn}\nabla_H^2\phi'. \tag{53}$$

Here ∇_H^2 is the two-dimensional Laplacian operator on the horizontal plane.

The differential Eqs. (53), (50), (51) and the boundary conditions (52) constitute a linear boundary-value problem that can be solved using the method of normal modes.

We write

$$(w', T', \phi') = [W(z), \Theta(z), \Phi(z)] \exp(st + ilx + imy), \tag{54}$$

and substitute into the differential equations to obtain

$$(D^2 - \alpha^2)W + \mathrm{Ra}\alpha^2\Theta - \mathrm{Rn}\alpha^2\Phi = 0, \tag{55}$$

$$W + \left(D^2 + \frac{N_B}{\mathrm{Le}}D - \frac{2N_AN_B}{\mathrm{Le}}D - \alpha^2 - s\right)\Theta - \frac{N_B}{\mathrm{Le}}D\Phi = 0, \tag{56}$$

$$\frac{1}{\varepsilon}W - \frac{N_A}{\mathrm{Le}}(D^2 - \alpha^2)\Theta - \left(\frac{1}{\mathrm{Le}}(D^2 - \alpha^2) - \frac{s}{\sigma}\right)\Phi = 0, \tag{57}$$

$$W = 0, \quad \Theta = 0, \quad \Phi = 0 \quad \text{at} \quad z = 0 \quad \text{and at } z = 1, \tag{58}$$

where

$$D \equiv \frac{d}{dz} \quad \text{and} \quad \alpha = (l^2 + m^2)^{1/2}. \tag{59}$$

Thus α is a dimensionless horizontal wavenumber.

For neutral stability the real part of s is zero. Hence we now write $s = i\omega$, where ω is real and is a dimensionless frequency.

We now employ a Galerkin-type weighted residuals method to obtain an approximate solution to the system of Eqs. (55)–(58). We choose as trial functions (satisfying the boundary conditions)

$$W_p = \Theta_p = \Phi_p = \sin p\pi z; \quad p = 1, 2, 3, \ldots \tag{60}$$

write

$$W = \sum_{p=1}^{N} A_pW_p, \quad \Theta = \sum_{p=1}^{N} B_p\Theta_p, \quad \Phi = \sum_{p=1}^{N} C_p\Phi_p, \tag{61}$$

substitute into Eqs. (55)–(57), and make the expressions on the left-hand sides of those equations (the residuals) orthogonal to the trial functions, thereby obtaining a system of $3N$ linear algebraic equations in the $3N$ unknowns $A_p, B_p, C_p; p = 1, 2, \ldots, N$. The vanishing of the determinant of coefficients produces the eigenvalue equation for the system. One can regard Ra as the eigenvalue. Thus Ra is found in terms of the other parameters.

5. Results and discussion

5.1. Non-oscillatory convection

First, we consider the case of non-oscillatory instability, when $\omega = 0$.

For a first approximation we take $N = 1$. This produces the result

$$\mathrm{Ra} = \frac{(\pi^2 + \alpha^2)^2}{\alpha^2} - \left(N_A + \frac{\mathrm{Le}}{\varepsilon}\right)\mathrm{Rn}. \tag{62}$$

Finding the minimum as α varies results in

$$\mathrm{Ra} = 4\pi^2 - \left(N_A + \frac{\mathrm{Le}}{\varepsilon}\right)\mathrm{Rn}, \tag{63}$$

with the minimum being attained at $\alpha = \pi$. We recognize that in the absence of nanoparticles we recover the well-known result that the critical Rayleigh number is equal to $4\pi^2$. Usually when one employs a single-term Galerkin approximation in this context one gets an overestimate by about 3% (e.g. 1750 instead of 1708 in the case of the standard Bénard problem) but in this case the approximation happens to give the exact result.

As we have noted, for a typical nanofluid Le is of order 10^2–10^3 and N_A is not much greater than 10. Hence the coefficient of Rn in Eq. (63) is large and negative. Thus under the approximations we have made so far we have the result that the presence of nanoparticles lowers the value of the critical Rayleigh number, usually by a substantial amount, in the case when Rn is positive, that is when the basic nanoparticle distribution is a top-heavy one.

It will be noted that in Eq. (63) the parameter N_B does not appear. The instability is almost purely a phenomenon due to buoyancy coupled with the conservation of nanoparticles. It is independent of the contributions of Brownian motion and thermophoresis to the thermal energy equation. Rather, the Brownian motion and thermophoresis enter to produce their effects directly into the equation expressing the conservation of nanoparticles so that the temperature and the particle density are coupled in a particular way, and that results in the thermal and concentration buoyancy effects being coupled in the same way. It is useful to emphasize this by rewriting Eq. (63) in the form

$$\mathrm{Ra} + \left(\frac{Le}{\varepsilon} + N_A\right)\mathrm{Rn} = 4\pi^2, \tag{64}$$

and noting that the left-hand side is the linear combination of the thermal Rayleigh number Ra and the concentration Rayleigh number Rn. The problem is analogous to the double-diffusive problem discussed in Section 9.1.1 of Nield and Bejan [14]. It is also analogous to the bioconvection problem discussed by Kuznetsov and Avramenko [15].

We have defined Rn in a way so that it is positive when the applied particle density increases upwards (the destabilizing situation). We note that Ra takes a negative value when Rn is sufficiently large. In this case the destabilizing effect of concentration is so great that the bottom of the fluid layer must be cooled relative to the top in order to produce a state of neutral stability.

We emphasize that the simple expression in Eq. (63) arises because the Lewis number has been assumed to be large. In order to estimate the contribution of the terms involving N_B we have investigated the two-term Galerkin results. The expression in the eigenvalue equation is complicated and it is difficult to make a statement that is simultaneously precise, simple and general. However, it is clear that the functions of N_B are of second degree. We conclude that for practical purposes Eq. (64) is a good approximation.

5.1.1. Oscillatory convection

We now consider the case $\omega \neq 0$. We confine ourselves to the one-term Galerkin approximation. The eigenvalue equation now takes the form

$$\mathrm{Ra}\alpha^2\left(\frac{J}{\mathrm{Le}} + \frac{i\omega}{\sigma}\right) + \mathrm{Rn}\alpha^2\left(\frac{N_AJ}{\mathrm{Le}} + \frac{J + i\omega}{\varepsilon}\right) = J(J + i\omega)\left(\frac{J}{\mathrm{Le}} + \frac{i\omega}{\sigma}\right), \tag{65}$$

where for shorthand we have written

$$J = \pi^2 + \alpha^2. \tag{66}$$

The real and imaginary parts of Eq. (65) yield

$$\frac{\mathrm{Ra}\alpha^2}{\mathrm{Le}} + \mathrm{Rn}\alpha^2\left(\frac{N_A}{\mathrm{Le}} + \frac{1}{\varepsilon}\right) = \frac{J^2}{\mathrm{Le}} - \frac{\omega^2}{\sigma}, \tag{67}$$

$$\omega\left\{\frac{\mathrm{Ra}\alpha^2}{\sigma} + \frac{\mathrm{Rn}\alpha^2}{\varepsilon} - \frac{J^2}{\mathrm{Le}} - \frac{J^2}{\sigma}\right\} = 0. \tag{68}$$

Again the critical value of α is found to be π. Hence one obtains the results

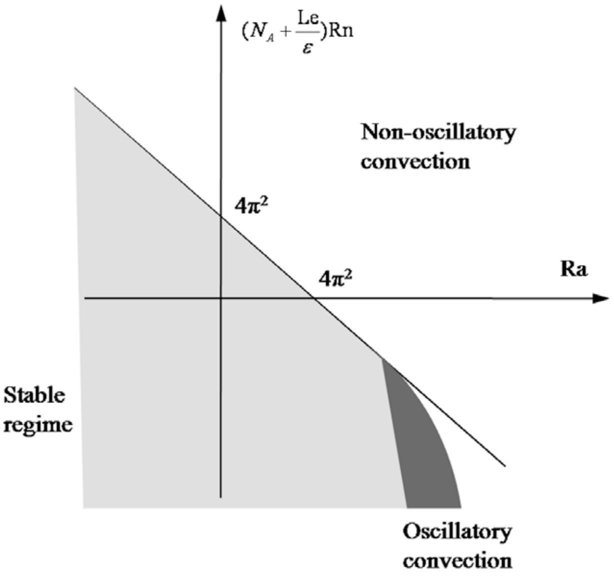

Fig. 1. Sketch of the stability and instability domains.

$$\frac{\text{Ra}}{\sigma} + \frac{\text{Rn}}{\varepsilon} = 4\pi^2\left(\frac{1}{\text{Le}} + \frac{1}{\sigma}\right), \tag{69}$$

$$\frac{\text{Le}\,\omega^2}{\pi^2\sigma} = 4\pi^2 - \left[\text{Ra} + \text{Rn}\left(N_A + \frac{\text{Le}}{\varepsilon}\right)\right]. \tag{70}$$

In order for ω to be real it is necessary that

$$\text{Ra} + \text{Rn}\left(N_A + \frac{\text{Le}}{\varepsilon}\right) \leqslant 4\pi^2. \tag{71}$$

Hence Eq. (69) gives the oscillatory stability boundary when Eq. (71) holds, and the angular frequency ω of the oscillation is given by Eq. (70). A sketch of $\text{Rn}(N_A + \text{Le}/\varepsilon)$ versus Ra is given in Fig. 1. The sketch is made on the assumption that $(\varepsilon N_A + Le)/\sigma$ is greater than unity. If that inequality is reversed than the labels on the axes need to be swapped around. The stability diagram is qualitatively similar to Fig. 9.2 in [14] which pertains to the double-diffusive Horton–Rogers–Lapwood problem.

There appears to be a qualitative discrepancy between our results and Fig. 4(b) in Tzou [6,7]. This figure indicates that the analysis in Tzou [6,7] leads to the prediction that the critical Rayleigh number is reduced by a substantial amount in the bottom-heavy case, whereas our analysis leads to a predicted increase in the value of the critical Rayleigh number for non-oscillatory instability in this case. Tzou offers no physical explanation for the substantial reduction. Tzou [6,7] uses the symbol Le to denote a Lewis number divided by the nanoparticle fraction decrement rather than a regular Lewis number. This means that his parameter Le tends to infinity as the nanoparticle fraction decrement tends to zero, i.e. in the limit as the nanofluid is replaced by a regular fluid. Accordingly, we hypothesize that it is possible that the solution obtained by Tzou [6,7] may become singular in some sense in this limit.

6. Conclusions

We have studied analytically using linear instability theory the onset of convection in a horizontal layer of a porous medium saturated by a nanofluid, employing a model used for the nanofluid that incorporates the effects of Brownian motion and thermophoresis. We found that for a typical nanofluid (for which the Lewis number is large) the primary contribution of the nanoparticles is via a buoyancy effect coupled with the conservation of nanoparticles, with the contribution of nanoparticles to the thermal energy equation being a second-order effect.

In this pioneering paper, we have employed a Darcy model for the momentum equation. We do not anticipate that the inclusion of a Brinkman term in that equation will have a major qualitative effect. Rather, the expected result would be that the value $4\pi^2$ is replaced by a larger value Ra_0 that depends on the hydrodynamic boundary conditions and increases with increase of the Darcy number. A consequence of the increase in Ra_0 is that the change in the value of Ra, for a given value of Rn, decreases as a percentage of the value of Ra_0. Thus, for example, a change from free–free boundary conditions to the more restrictive rigid–rigid boundary conditions, something that increases the value of Ra_0, leads to a decrease in the sensitivity of Ra to a given change in Rn.

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